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HYDRODYNAMICAL INTERACTION OF TWO PARTICLES COVERED WITH A POROUS LAYER

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Abstract—The force acting on two particles covered with a porous layer which move at a finite relative velocity along the line of minimal separation was calculated. The fluid flow in the porous layer was modeled by the Brinkman equations. The force is found as the leading term of an asymptotic expansion valid when the gap between particles is much less than the smallest principal radius of its curvature. The solution has no singularity when the surfaces are almost in contact (i.e. the gel layers are touching), which differs from the case of smooth surfaces.

Key Words: gel layer, particles, porosity, interaction

INTRODUCTION

The slow motion of solids in a viscous fluid has been considered by many authors (Goldman *et al.* 1967; O'Neil 1964; Dean & O'Neil 1963; Cox 1974). Goldman *et al.* (1967) applied lubrication theory to obtain approximate solutions for such motions. This theory was extended by Cox (1974) to find the forces and torques acting on any two smooth solid surfaces separated by a viscous fluid, the surfaces being such that if they were brought together the contact would occur at a single point only. The results are obtained as the leading terms of an asymptotic expansion valid when the gap between the particles is much less than the smallest principal radius of its curvature. The solution has a singularity when the particles are in contact.

Experimental investigations (Churaev *et al.* 1981; Kim & Anderson 1989) have shown that some porous structures created, for example, by adsorbed or terminally attached polymers, exist on the solid surfaces of many colloidal particles. Such porous layers could also be formed by the dissolution of the particle surface itself in a leaching operation. The presence of a porous layer on the surface, called a "gel layer", affects the hydrodynamic phenomena (Garvey *et al.* 1975; Cohen *et al.* 1984). Fluid flow within a porous layer is often modeled by the Brinkman equation, which applies when inertial effects are negligible.

In this paper lubrication theory is used to investigate the effects of porous gel layers on the hydrodynamic interaction between two particles approaching each other in the direction of the line connecting the points of minimal separation. It is assumed that their contact may occur only at a single point on the surface of the gel layer and the gel layers do not deform. It is assumed that the density and other properties of the gel layers do not depend on the distance from the particle surface. It is known (Cosgrove 1990) that the density of the adsorbed layers of some polymers is not uniform in the direction normal to the solid surface. Nevertheless, the solution obtained allows one to estimate the influence of adsorbed polymer layers on the hydrodynamical behavior of a colloid system.

The force acting on the particles is found as the leading term of an asymptotic expansion valid for a small gap between the particles. The solution obtained has no singularity at the point of contact. If the thickness of the gel layer tends to zero the results reduce to those given by Cox (1974).

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STATEMENT OF THE PROBLEM

Consider two particles covered with a gel layer of thickness $\tilde{\delta}$ suspended in a viscous fluid. It is assumed that the particles are approaching each other in the normal direction (figure 1). The Reynolds number (Re) is assumed to be so small that inertial effects may be neglected. The surfaces of the particles are W and W', respectively. The fluid, of viscosity η, in the gap between the particles is incompressible. The objective is to find the viscous force acting on the surfaces of the moving particles as a result of their motion. Following the nomenclature of Cox (1974), we define the local Cartesian axes $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ with origin O, which lies on the surface W of one of the particles at the point of minimum separation of the surfaces. The axis $\tilde{x}_3$ is taken to be normal to the surface W and, hence, to the surface W' of the other particle. The axes $\tilde{x}_1$ and $\tilde{x}_2$ are tangential to the surface W and lie in the directions of the principal curvatures of the surface W at the point O. The surface W may be written for the small values of $\tilde{r} = (\tilde{x}_1 + \tilde{x}_2)^{1/2}$ as

$$\tilde{x}_3 = -\tilde{h}_1(\tilde{x}_1, \tilde{x}_2) + O\left(\frac{\tilde{r}^2}{L^2}\right), \tag{1}$$

where

$$\tilde{h}_1(\tilde{x}_1, \tilde{x}_2) = \frac{\tilde{x}_1^2}{2\tilde{R}_1} + \frac{\tilde{x}_2^2}{2\tilde{R}_2}. \tag{2}$$

$\tilde{R}_1$ and $\tilde{R}_2$ are the principal radii of curvature of the surface W at the point O; $L = \min\{\tilde{R}_1, \tilde{R}_2, \tilde{S}_1, \tilde{S}_2\}$, $\tilde{S}_1$ and $\tilde{S}_2$ are the principal radii of curvature of the surface W' at the point O′; O′(0, 0, a), a is the gap width between the solid surfaces of the particles ($a \geqslant 2\tilde{\delta}$). The surface W' may be written in the form

$$\tilde{x}_3 = a + \tilde{h}_2(\tilde{x}_1, \tilde{x}_2) + O\left(\frac{\tilde{r}^2}{L^2}\right), \tag{3}$$

where

$$\tilde{h}_2(\tilde{x}_1, \tilde{x}_2) = \tilde{x}_1^2\left(\frac{\cos^2\theta}{2\tilde{S}_1} + \frac{\sin^2\theta}{2\tilde{S}_2}\right) + \tilde{x}_1\tilde{x}_2\left(\frac{1}{\tilde{S}_1} - \frac{1}{\tilde{S}_2}\right)\sin\theta\cos\theta + \tilde{x}^2\left(\frac{\cos^2\theta}{2\tilde{S}_1} + \frac{\sin^2\theta}{2\tilde{S}_2}\right); \tag{4}$$

θ is the angle between the principal axes of the curvature of W and W' at the point O′.

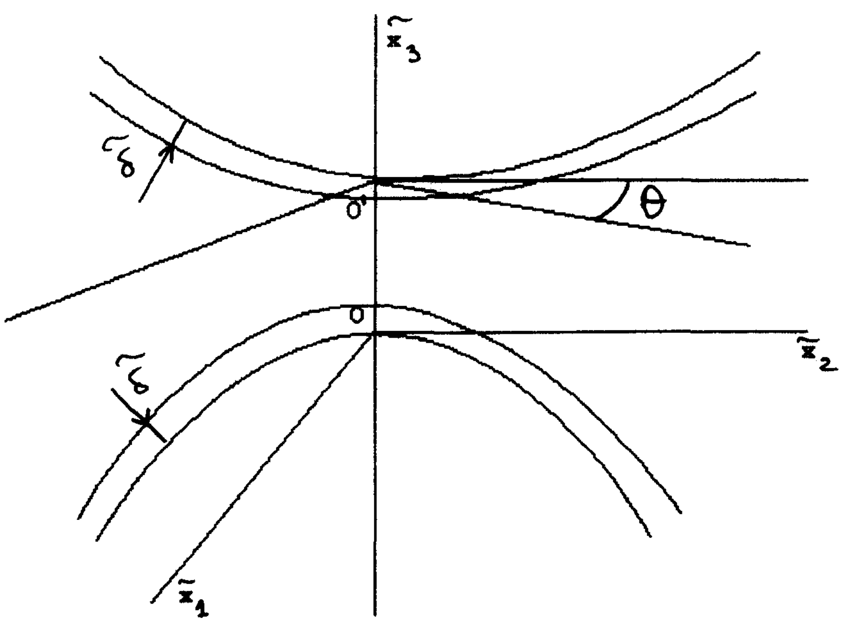

Figure 1. The system geometry.

The fluid flow in the gap between the particles satisfies the Stokes equations

$$\eta\,\Delta\tilde{\mathbf{u}} - \nabla\tilde{p} = 0 \qquad [5]$$

and

$$\operatorname{div}\tilde{\mathbf{u}} = 0, \qquad [6]$$

where $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ and $\tilde{p}$ are the fluid velocity and pressure, respectively, related to the $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ axis system; and η is the fluid viscosity. The region of the porous layer may be defined as

$$-\tilde{h}_1 \leqslant \tilde{x}_3 \leqslant -\tilde{h}_1 + \tilde{\delta} \quad \text{and} \quad a + \tilde{h}_2 - \tilde{\delta} \leqslant \tilde{x}_3 \leqslant \tilde{a} + \tilde{h}_2. \qquad [7]$$

For analysis of flow within a porous layer, we use the Brinkman equations

$$\eta^*\,\Delta\tilde{\mathbf{u}}^* - \nabla\tilde{p}^* - k\tilde{\mathbf{u}}^* = 0 \qquad [8]$$

and

$$\operatorname{div}\tilde{\mathbf{u}}^* = 0, \qquad [9]$$

where $\tilde{\mathbf{u}}^* = (\tilde{u}_1^*, \tilde{u}_2^*, \tilde{u}_3^*)$, $\tilde{p}^*$ and η^* are the velocity, pressure and fluid viscosity within the porous layer; k is the hydrodynamic screening parameter which may be calculated using the Carman relation (Kim & Russel 1985), or another model (Anderson *et al.* 1991). All functions related to the porous layer are denoted with an asterisk. The no-slip boundary conditions at the walls W and W' are

$$\tilde{\mathbf{u}}^* = \tilde{\mathbf{U}}_1 \quad \text{on } W \qquad [10]$$

and

$$\tilde{\mathbf{u}}^* = \tilde{\mathbf{U}}_2 \quad \text{on } W', \qquad [11]$$

where $\tilde{\mathbf{U}}_i = (0, 0, \tilde{U}_i)$, $i = 1, 2$.

Continuity of the stress tensor $\tilde{\sigma}$ and the velocity field is assumed at the gel layer/fluid interface:

$$\tilde{\sigma}_{ij} n_j = \tilde{\sigma}_{ij}^* n_j; \qquad [12]$$

here $\mathbf{n}$ is the unit normal to the surface W and

$$\tilde{\mathbf{u}} = \tilde{\mathbf{u}}^*. \qquad [13]$$

It is also assumed that the pressure tends to zero at infinity:

$$\tilde{p} \to 0, \quad \tilde{r} \to \infty. \qquad [14]$$

The task is to solve the set of hydrodynamic equations [5], [6], [8] and [9] subject to the boundary conditions [10]–[14] for $\tilde{\mathbf{u}}$, $\tilde{p}$ and $\tilde{\mathbf{u}}^*$, $\tilde{p}^*$; and then to compute the force of the hydrodynamic interaction between two particles covered with the porous layer.

METHOD OF SOLUTION

In order to solve the set of hydrodynamic equations [5], [6], [8] and [9] subject to boundary conditions [10]–[14], it is convenient to use the following dimensionless variables (x_1, x_2, x_3):

$$x_1 = \frac{\tilde{x}_1}{\sqrt{La}}, \quad x_2 = \frac{\tilde{x}_2}{\sqrt{La}}, \quad x_3 = \frac{\tilde{x}_3}{a}. \qquad [15]$$

With these definitions, the surface W becomes

$$x_3 = -h_1(x_1, x_2) + O(\epsilon) \qquad [16]$$

and the surface W' becomes

$$x_3 = 1 + h_2(x_1, x_2) + O(\epsilon), \qquad [17]$$

where the functions $h_1(x_1, x_2)$ and $h_2(x_1, x_2)$ are defined as

$$h_1(x_1, x_2) = \frac{x_1^2}{2R_1} + \frac{x_2^2}{2R_2} \quad [18]$$

and

$$h_2(x_1, x_2) = x_1^2\left(\frac{\cos^2\theta}{2S_1} + \frac{\sin^2\theta}{2S_2}\right) + x_1x_2\left(\frac{1}{S_1} - \frac{1}{S_2}\right)\sin\theta\cos\theta + x_2^2\left(\frac{\cos^2\theta}{2S_2} + \frac{\sin^2\theta}{2S_1}\right). \quad [19]$$

$R_i = \tilde{R}_i/L$ and $S_i = \tilde{S}_i/L$ ($i = 1, 2$) are the dimensionless principal radii of curvature; and $\epsilon = \sqrt{a/L}$ is a parameter which is assumed to be very small.

The following dimensionless variables and parameters will be used:

$$p = \frac{\tilde{p}}{p_0}, \qquad u_1 = \epsilon\frac{\tilde{u}_1}{u_0}, \qquad u_2 = \epsilon\frac{\tilde{u}_2}{u_0}, \qquad u_3 = \frac{\tilde{u}_3}{u_0},$$

$$\bar{p} = \frac{\tilde{p}^*}{p_0}, \qquad \bar{u}_1 = \epsilon\frac{\tilde{u}_1^*}{u_0}, \qquad \bar{u}_2 = \epsilon\frac{\tilde{u}_2^*}{u_0}, \qquad \bar{u}_3 = \frac{\tilde{u}_3^*}{u_0},$$

$$U_i = \frac{\tilde{U}_i}{u_0} \quad (i = 1, 2), \qquad p_0 = \frac{u_0\eta L}{a^2}, \qquad u_0 = \tilde{U}_2 - \tilde{U}_1, \qquad \alpha = \frac{\eta^*}{\eta},$$

$$R_i = \frac{\tilde{R}_i}{L}, \quad S_i = \frac{\tilde{S}_i}{L} \quad (i = 1, 2), \quad \tilde{r} = (\tilde{x}_1 + \tilde{x}_2)^{1/2}$$

and

$$\omega = \tilde{\delta}\sqrt{\frac{k}{\eta}}, \quad \gamma = \frac{\tilde{\delta}}{a}, \quad r = \frac{\tilde{r}}{\sqrt{La}}, \quad [20]$$

where u_0 and p_0 are the characteristic scales of the velocity and pressure.

With the dimensionless variables ([15] and [20]), then [5] and [6] may be rewritten in the form:

$$\left(\epsilon^2\frac{\partial^2}{\partial x_1^2} + \epsilon^2\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)u_i - \frac{\partial p}{\partial x_i} = 0 \quad (i = 1, 2), \quad [21]$$

$$\epsilon^2\left(\epsilon^2\frac{\partial^2}{\partial x_1^2} + \epsilon^2\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)u_3 - \frac{\partial p}{\partial x_3} = 0 \quad [22]$$

and

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0. \quad [23]$$

The region of the porous layer takes the form:

$$-h_1 \leqslant x_3 \leqslant -h_1 + \gamma \quad \text{and} \quad 1 + h_2 - \gamma \leqslant x_3 \leqslant 1 + h_2. \quad [24]$$

With the dimensionless variables, the Brinkman equations [8] and [9] become:

$$\alpha\left(\epsilon^2\frac{\partial^2}{\partial x_1^2} + \epsilon^2\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\bar{u}_i - \frac{\partial\bar{p}}{\partial x_i} - \frac{\omega^2}{\gamma^2}\bar{u}_i = 0 \quad (i = 1, 2), \quad [25]$$

$$\alpha\epsilon^2\left(\epsilon^2\frac{\partial^2}{\partial x_1^2} + \epsilon^2\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\bar{u}_3 - \frac{\partial\bar{p}}{\partial x_3} - \frac{\epsilon^2\omega^2}{\gamma^2}\bar{u}_3 = 0 \quad [26]$$

and

$$\frac{\partial\bar{u}_1}{\partial x_1} + \frac{\partial\bar{u}_2}{\partial x_2} + \frac{\partial\bar{u}_3}{\partial x_3} = 0. \quad [27]$$

The no-slip boundary conditions expressed in the dimensionless variables are

$$\left.\begin{array}{l} \bar{u}_i = 0 \ (i = 1, 2) \quad \text{on} \quad W \text{ and } W' \\ \bar{u}_3 = U_i \text{ on } W \quad \text{and} \quad \bar{u}_3 = U_2 \text{ on } W'. \end{array}\right\} \quad [28]$$

Continuity of the stress tensor and the velocity field on the porous layer/fluid interface may be written as

$$\sigma_{ij} n_j = \sigma^*_{ij} n_j, \quad [29]$$

where $\mathbf{n} = \{x_1\epsilon/R_1, x_2\epsilon/R_2, 1 + O(\epsilon)\}$ is the unit normal to the surface W; σ is the dimensionless stress tensor; and

$$u_i = u^*_i \quad (i = 1, 2, 3). \quad [30]$$

The boundary condition for the pressure takes the form

$$p \to 0, \quad r \to \infty, \quad [31]$$

where $r = (x_1 + x_2)^{1/2}$.

Assume that the velocity and pressure may be expanded in powers of the parameter ϵ in the following form:

$$\mathbf{u} = \sum_i u_i \epsilon^i, \quad p = \sum_i p_i \epsilon^i, \quad \bar{\mathbf{u}} = \sum_i \bar{u}_i \epsilon^i, \quad \bar{p} = \sum_i \bar{p}_i \epsilon^i. \quad [32]$$

Consider only the leading terms of expansions [32], the terms of order $O(\epsilon)$ will be neglected (further, indices will be dropped).

It is convenient to change variables to (y_1, y_2, y_3) and use the local variable ξ in the region of the porous layer:

$$y_1 = x_1, \quad y_2 = x_2, \quad y_3 = x_3 + h(x_1, x_2); \quad \xi = \frac{x_3 + h(x_1, x_2)}{\gamma}. \quad [33]$$

The surface W expressed in the new variables becomes

$$y_3 = 0 \quad \text{at } \xi = 0 \quad [34]$$

and the surface W' becomes

$$y_3 = h(y_1, y_2) \quad \text{at } \xi = \frac{h(y_1, y_2)}{\gamma}, \quad [35]$$

where

$$h(y_1, y_2) = 1 + y_1^2\left(\frac{\cos^2\theta}{2S_1} + \frac{\sin^2\theta}{2S_2} + \frac{1}{2R_1}\right) + y_1 y_2\left(\frac{1}{S_1} - \frac{1}{S_2}\right)\sin\theta\cos\theta + y_2^2\left(\frac{\cos^2\theta}{2S_2} + \frac{\sin^2\theta}{2S_1} + \frac{1}{2R_2}\right). \quad [36]$$

Neglecting the terms of order $O(\epsilon)$, [21]–[23] take the form

$$\frac{\partial^2}{\partial y_3^2} u_i - \frac{\partial p}{\partial y_i} = 0 \quad (i = 1, 2), \quad [37]$$

$$\frac{\partial p}{\partial x_3} = 0 \quad [38]$$

and

$$\frac{\partial u_1}{\partial y_1} + \frac{\partial u_2}{\partial y_2} + \frac{\partial u_3}{\partial y_3} + \frac{y_1}{R_1}\frac{\partial u_1}{\partial y_3} + \frac{y_2}{R_2}\frac{\partial u_2}{\partial y_3} = 0; \quad [39]$$

and [25]–[27] take the form

$$\frac{\partial^2}{\partial \xi^2}\bar{u}_i - \frac{\gamma^2}{\beta^2}\frac{\partial \bar{p}}{\partial x_i} - \frac{\omega^2}{\beta^2}\bar{u}_i = 0 \quad (i = 1, 2), \quad [40]$$

$$\frac{\partial \bar{p}}{\partial \xi} = 0 \quad [41]$$

and

$$\gamma \frac{\partial \bar{u}_1}{\partial y_1} + \gamma \frac{\partial \bar{u}_2}{\partial y_2} + \frac{\partial \bar{u}_3}{\partial \xi} + \frac{y_1}{R_1}\frac{\partial \bar{u}_1}{\partial \xi} + \frac{y_2}{R_2}\frac{\partial \bar{u}_2}{\partial \xi} = 0, \qquad [42]$$

where $\alpha^2 = \beta$.

The boundary conditions on W and W' expressed in the new variables are

$$\left.\begin{aligned} &\bar{u}_i = 0\ (i = 1, 2), \quad \bar{u}_3 = U_1 \text{ at } \xi = 0 \\ &\bar{u}_i = 0\ (i = 1, 2), \quad \bar{u}_3 = U_2 \text{ at } \xi = \frac{h}{\gamma} \end{aligned}\right\}. \qquad [43]$$

Neglecting the terms of order $O(\epsilon)$, from [30] and [29] it follows that at the porous layer/liquid interface the boundary conditions have the form

$$p = \bar{p}, \qquad [44]$$

$$u_i|_{y_3=\gamma} = \bar{u}_i|_{\xi=1}, \quad u_i|_{y_3=h-\gamma} = \bar{u}_i|_{\xi=h/\gamma-1} \qquad [45]$$

and

$$\frac{\partial}{\partial y_3} u_i|_{y_3=\gamma} = \frac{\alpha}{\gamma}\frac{\partial}{\partial z}\bar{u}_i|_{\xi=1}, \quad \frac{\partial}{\partial y_3} u_i|_{y_3=h-\gamma} = \frac{\alpha}{\gamma}\frac{\partial}{\partial z}\bar{u}_i|_{\xi=h/\gamma-1}. \qquad [46]$$

The problem is thus reduced to solving [37]–[42] subject to boundary conditions [43]–[46].

From [38] and [41] it follows that the pressure within and outside the porous layer is a function of y_1 and y_2. The solution of [40], which satisfies the no-slip condition [43] on W, has the form

$$\bar{u}_i = 2C^i \sinh\left(\frac{\omega\xi}{\beta}\right) + \left(\frac{\gamma}{\omega}\right)^2 \frac{\partial p}{\partial y_i}\left[\exp\left(\frac{-\omega\xi}{\beta}\right) - 1\right] \quad (i = 1, 2), \qquad [47]$$

where C^i is the constant obtained from the ξ integration and thus an arbitrary function of y_1 and y_2.

The solution of [37], obtained from the y_3 integration, has the form

$$u_i = \tfrac{1}{2} y_3^2 \frac{\partial p}{\partial y_i} + A_i y_3 + B_i, \qquad [48]$$

where the quantities $A_i = A_i(y_1, y_2)$ and $B_i = B_i(y_i, y_2)$ are determined from the boundary conditions [45] and [46] with the result

$$A_i = C^i F + \frac{\partial p}{\partial y_i} G, \quad B_i = C^i S + \frac{\partial p}{\partial y_i} Q, \qquad [49]$$

where

$$F = 2\frac{\omega\beta}{\gamma}\cosh\frac{\omega}{\beta}, \qquad G = -\left[\frac{\beta\gamma}{\omega}\exp\left(-\frac{\omega}{\beta}\right) + \gamma\right]$$

$$S = 2\sinh\left(\frac{\omega}{\beta}\right) - 2\omega\beta\cosh\left(\frac{\omega}{\beta}\right), \quad Q = \left(\frac{\gamma}{\omega}\right)^2\left[\exp\left(-\frac{\omega}{\beta}\right) - 1\right] + \tfrac{1}{2}\gamma^2 + \frac{\beta\gamma^2}{\omega}\exp\left(-\frac{\omega}{\beta}\right). \qquad [50]$$

Substituting the expressions for $\bar{u}_1$ and $\bar{u}_2$ given by [47] into [42] and integrating with respect to ξ, subject to the no-slip condition [43] on W, one obtains

$$\begin{aligned} \bar{u}_3 = &-\gamma\left[\left(\frac{\gamma}{\omega}\right)^2 \Delta p\left\{-\frac{\beta}{\omega}\left[\exp\left(-\frac{\omega\xi}{\beta}\right) - 1\right] - \xi\right\} + \frac{2\beta}{\omega}\left[\cosh\left(\frac{\omega\xi}{\beta}\right) - 1\right]\left(\frac{\partial C^1}{\partial y_1} + \frac{\partial C^2}{\partial y_2}\right)\right] \\ &- \frac{\partial h}{\partial y_1}\left[2C^1\sinh\left(\frac{\omega\xi}{\beta}\right) + \left(\frac{\gamma}{\omega}\right)^2\frac{\partial p}{\partial y_1}\right]\left[\exp\left(-\frac{\omega\xi}{\beta}\right) - 1\right] - \frac{\partial h}{\partial y_2}\left[2C^2\sinh\left(\frac{\omega\xi}{\beta}\right) + \left(\frac{\gamma}{\omega}\right)^2\frac{\partial p}{\partial y_2}\right] \\ &\times\left[\exp\left(-\frac{\omega\xi}{\beta}\right) - 1\right] + U_1, \end{aligned} \qquad [51]$$

where $h = h(y_1, y_2)$ is given by [36].

Substitution of the expressions for u_1 and u_2 given by [48] into [39] and the y_3 integration, subject to the continuity of the velocity on the layer/fluid interface [45], yields

$$u_3 = \Delta p\left[\!\!\left[-\tfrac{1}{6}(y_3^3 - \gamma^3) + \frac{\gamma^3}{\omega^2}\left\{\frac{\beta}{\omega}\left[\exp\left(-\frac{\omega}{\beta}\right) - 1\right] + 1\right\} - \tfrac{1}{2}G(y_3^3 - y^3) - Q(y_3 - \gamma)\right]\!\!\right]$$
$$+ \left(\frac{\partial C^1}{\partial y_1} + \frac{\partial C^2}{\partial y_2}\right)\left\{2\frac{\beta}{\omega}\left[\cosh\left(\frac{\omega}{\beta}\right) - 1\right] - \tfrac{1}{2}F(y_3^2 - \gamma^2) - S(y_3 - \gamma)\right\}$$
$$- \frac{\partial h}{\partial y_1}\left(\tfrac{1}{2}y_3^2\frac{\partial p}{\partial y_1} + A_1 y_3 + B_1\right) - \frac{\partial h}{\partial y_2}\left(\tfrac{1}{2}y_3^2\frac{\partial p}{\partial y_2} + A_2 y_3 + B_2\right) + W_1, \qquad [52]$$

where A_i and B_i $(i = 1, 2)$ are given by [49].

In order to compute the pressure, it is necessary to obtain the expressions for the velocities in the porous layer near the surface W'. As before, we solve [37]–[42] subject to the boundary conditions [43]–[46] near the surface W' and obtain

$$\bar{u}_i = \frac{\partial p}{\partial y_i}\left[\!\!\left[2E(h)\sinh\left[\frac{\omega}{\beta}\left(\xi - \frac{h}{\gamma}\right)\right] + \left(\frac{\gamma}{\omega}\right)^2\left\{\exp\left[-\frac{\omega}{\beta}\left(\xi - \frac{h}{\gamma}\right)\right] - 1\right\}\right]\!\!\right],$$

$$\bar{u}_3 = -\gamma\,\Delta p\left(2T(h)\frac{\beta}{\omega}\left\{\cosh\left[\frac{\omega}{\beta}\left(\xi - \frac{h}{\gamma}\right)\right] - 1\right\} - \left(\frac{\gamma}{\omega}\right)^2\left[\!\!\left[\frac{\beta}{\omega}\left\{\exp\left[-\frac{\omega}{\beta}\left(\xi - \frac{h}{\gamma}\right)\right] - 1\right\} + \xi - \frac{h}{\gamma}\right]\!\!\right]\right)$$
$$= -\gamma\left(\frac{\partial p}{\partial y_1}\frac{\partial h}{\partial y_1} + \frac{\partial p}{\partial y_2}\frac{\partial h}{\partial y_2}\right)\left[\!\!\left[2\frac{\beta}{\omega}\frac{dT}{dh}\left\{\cosh\left[\frac{\omega}{\beta}\left(\xi - \frac{h}{\gamma}\right)\right] - 1\right\} - \frac{2T}{\gamma}\sinh\left[\frac{\omega}{\beta}\left(\xi - \frac{h}{\gamma}\right)\right]\right.$$
$$\left. - \frac{\gamma}{\omega^2}\left\{\exp\left[-\frac{\omega}{\beta}\left(\xi - \frac{h}{\gamma}\right)\right] - 1\right\}\right]\!\!\right] - \frac{\partial h}{\partial y_1}\bar{u}_1 - \frac{\partial h}{\partial y_2}\bar{u}_2 + U_2 \qquad [53]$$

and

$$C^i = \Gamma(h)\frac{\partial p}{\partial y_i}, \qquad [54]$$

where

$$T(h) = E(h) + \Gamma(h), \qquad [55]$$

h is the function of y_1 and y_2 given by [43] and

$$\Gamma(h) = -\left\{E(h)\sinh\left(\frac{\omega}{\beta}\right) - \left(\frac{\gamma}{\omega}\right)^2\left[\exp\left(\frac{\omega}{\beta}\right) - 1\right] + \tfrac{1}{2}(h-1)^2 + (h-\gamma)G + Q\right\}$$
$$\times\left[2\sinh\left(\frac{\omega}{\beta}\right) + F(h-1) + S\right]^{-1},$$

$$E(h) = \frac{\gamma}{\beta\omega}\,\frac{h - 2\gamma + 2\dfrac{\gamma\beta}{\omega}\sinh\left(\dfrac{\omega}{\beta}\right)}{2\cosh\left(\dfrac{\omega}{\beta}\right)}. \qquad [56]$$

Since $u_3 = \bar{u}_3$ on the bound of the porous layer, from [53] and [54], after simplifications, it follows that

$$\mathrm{div}_s[\Phi(h)\,\nabla_s p] = U_2 - U_1, \qquad [57]$$

where

$$\Phi(h) = -\tfrac{1}{12}[(h-\gamma)^3-\gamma^3] + \frac{\gamma^3}{\omega^2}\left[\frac{\beta}{\omega}\exp\left(-\frac{\omega}{\beta}\right)+1\right] - 0.5G[(h-\gamma)^2-\gamma^2]$$

$$-(h-2\gamma)Q + \left\{\frac{2\beta}{\omega}\left(\cosh\frac{\omega}{\beta-1}\right) - 0.5[(h-\gamma)^2-\gamma^2]F - (h-2\gamma)S\right\}\Gamma(h)$$

$$+2T(h)\left(\frac{\beta\gamma}{\omega}\right)\left(\cosh\frac{\omega}{\beta-1}\right) - \frac{\gamma^3}{\omega^2}\left[\frac{\beta}{\omega}\exp\left(-\frac{\omega}{\beta}\right)-1\right],$$

$p(y_1, y_2)$ is an unknown pressure; $h(y_1, y_2)$ is defined by [36] and Φ is defined as a function of h and hence may be written as the function of y_1 and y_2. All derivatives are taken with respect to the variables y_1 and y_2.

It may be shown that the above result reduces to those given by Cox (1974) in the limit as the thickness of the porous layer approaches zero. In this case, the function $\Phi(h)$ reduces to $-h^3/12$.

Following Cox (1974), the expression for h may be written in the form

$$h = 1 + \mathbf{y}^{\mathrm{T}}\mathfrak{R}\mathbf{y}, \qquad [58]$$

where $\mathbf{y}$ is the column vector

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad [59]$$

and $\mathbf{y}^{\mathrm{T}}$ is its transpose. Then $\mathfrak{R}$ is the matrix

$$\mathfrak{R} = \begin{bmatrix} \dfrac{1}{2R_1} + \dfrac{\cos^2\theta}{2S_2} + \dfrac{\sin^2\theta}{2S_2} & \dfrac{\sin\theta\cos\theta}{2}\left(\dfrac{1}{S_1} - \dfrac{1}{S_2}\right) \\ \dfrac{\sin\theta\cos\theta}{2}\left(\dfrac{1}{S_1} - \dfrac{1}{S_2}\right) & \dfrac{1}{2R_2} + \dfrac{\sin^2\theta}{2S_1} + \dfrac{\cos^2\theta}{2S_2} \end{bmatrix}. \qquad [60]$$

Defining λ_1 and λ_2 as the eigenvalues, the expression for h [58] may be transformed into the form

$$h = 1 + \lambda_1\hat{y}_1^2 + \lambda_2\hat{y}_2^2. \qquad [61]$$

The variables $\hat{y}_1$ and $\hat{y}_2$ are defined with respect to the coordinate system where axes are chosen in the directions of the normalized eigenvectors of $\mathfrak{R}$.

Compute the function $p = p(y_1, y_2)$ as a function of h. In this case, [57] takes the form

$$(4\lambda_1\hat{y}_1^2 + 4\lambda_2\hat{y}_2^2)\frac{\mathrm{d}}{\mathrm{d}h}\left(\Phi\frac{\mathrm{d}}{\mathrm{d}h}p\right) + (2\lambda_1 + 2\lambda_2)\Phi\frac{\mathrm{d}}{\mathrm{d}h}p = U_1 - U_2. \qquad [62]$$

Note that if $\Phi(\mathrm{d}/\mathrm{d}h)p = \text{const}$, then p satisfies [62] and hence

$$(2\lambda_1 + 2\lambda_2)\Phi\frac{\mathrm{d}}{\mathrm{d}h}p = U_1 - U_2. \qquad [63]$$

The solution to [62] which satisfies the boundary condition [31] is

$$p = \frac{U_2 - U_1}{2(\lambda_1 + \lambda_2)}\int_{\infty}^{1}\frac{\mathrm{d}h}{\Phi(h)}. \qquad [64]$$

The force $\mathbf{F}$ exerted by the fluid on W is given by

$$F_i = \int_W \sigma_{ij}n_j\,\mathrm{d}s, \qquad [65]$$

where $\mathbf{n}$ is defined as the unit normal to the surface W from solid to fluid; σ_{ij} is the stress tensor corresponding to $\mathbf{u}$ and p; and $\mathrm{d}S$ is an element of area of the surface. For small values of $r = (\hat{y}_1 + \hat{y}_2)^{1/2}$, $\mathbf{n}$ and $\mathrm{d}S$ may be written as terms of order $O(1)$ with respect to ϵ in the form

$$\mathbf{n} = (0, 0, 1), \quad \mathrm{d}S = \mathrm{d}\hat{y}_1\,\mathrm{d}\hat{y}_2. \qquad [66]$$

It may be also shown that the stress tensor of order $O(1)$ is diagonal and the components p_{ii}: $p_{ii} = p(i = 1, 2, 3)$.

For the evaluation of the integral [65] the surface W is divided into two areas S_ρ and Σ_ρ, where the area S_ρ is part of W for which the vector **y** given by [65] satisfies the relation

$$\lambda_1 \hat{y}_1^2 + \lambda_2 \hat{y}_2^2 \leqslant \frac{\rho^2}{\epsilon^2}, \qquad [67]$$

where ρ is a small arbitrary parameter.

In order to integrate [65], the substitutions

$$\hat{y}_1 = \frac{r \cos \phi}{\sqrt{\lambda_1}}, \quad \hat{y}_2 = \frac{r \cos \phi}{\sqrt{\lambda_2}} \qquad [68]$$

are made so that [61] takes the form

$$h = 1 + r^2 \qquad [69]$$

and [67] may be written in the form

$$r \leqslant \frac{\rho}{\epsilon}. \qquad [70]$$

Since ρ is an arbitrary small parameter and $\epsilon \to 0$ when the distance between the surfaces tends to zero, it follows that

$$0 \leqslant r \leqslant \infty. \qquad [71]$$

The force **F** [65] may be written as

$$F_i = \int_{S_\rho} \sigma_{ij} n_j \, dS + \int_{\Sigma_\rho} \sigma_{ij} n_j \, dS. \qquad [72]$$

The second integral in [72] tends to a finite limit as the surfaces approach, because the parameter ρ is independent of ϵ. The first integral may be evaluated by expressing all quantities in r and ϕ variables. Performing the ϕ integration one finds

$$F_1 = F_2 = 0,$$

$$F_3 = -\frac{2\pi}{\sqrt{\lambda_1 \lambda_2}} \int_{r=0}^{\infty} pr \, dr. \qquad [73]$$

Since $h = 1 + r^2$, substituting [64] into [73] one obtains

$$F_3 = \frac{\pi(U_2 - U_1)}{2(\lambda_1 + \lambda_2)\sqrt{\lambda_1 \lambda_2}} \int_1^{\infty} \frac{(h-1)\, dh}{\Phi(h)}. \qquad [74]$$

Converting to dimensional variables, the force acting on the surface of the particle may be written in the form

$$\mathbf{F} = (0, 0, \mathscr{F}),$$

$$\mathscr{F} = \frac{\eta\pi(U_2^* - U_1^*)L^2}{2a(\lambda_1 + \lambda_2)\sqrt{\lambda_1 \lambda_2}} \int_1^{\infty} \frac{(h-1)\, dh}{\Phi(h)}, \qquad [75]$$

where $\Phi(h)$ is given by [57] and λ_1 and λ_2 are the eigenvalues of the matrix $\mathfrak{R}$ given by [60]. Note the integrals in the right-hand sides of [74] and [74] do not depend on the geometrical parameters of the system considered. This fact allows one to reduce use of the model developed for systems of different geometry to calculation of the eigenvalues λ_1, λ_2. It is easy to see, that in the case of two spheres of radii $\tilde{R}$ and $\tilde{S}$, respectively,

$$\lambda_1 = \lambda_2 = \frac{L}{2}\left(-\frac{1}{\tilde{R}} + \frac{1}{\tilde{S}}\right). \qquad [76]$$

Consideration of two crossed cylinders leads to

$$\lambda_1 = \lambda_2 = \frac{L}{8}\left(-\frac{2}{\tilde{R}} + \frac{2}{\tilde{S}}\right) + \left[\frac{L^2}{16\tilde{S}\tilde{R}}\sin^2\theta + \frac{L^2}{8}\left(\frac{2}{\tilde{R}} + \frac{2}{\tilde{S}}\right)^2\right]^{1/2},$$

where $\tilde{R}$ and $\tilde{S}$ are the cylinder radii and θ is the angle between the cylinder axes.

In the case when the thickness of the gel layer tends to zero the result reduces to that obtained by Cox (1974). However, the solution obtained has no singularity at a point of contact when the gel layer has finite thickness.

DISCUSSION OF RESULTS

The theory considered was applied to obtain the force acting on two particles covered with a porous layer which move at a finite relative velocity along the line of minimal separation. The fluid flow in the porous layer was modeled by the Brinkman equations. The force is found as the leading term of an asymptotic expansion valid when the gap between the particles is much less than the smallest principal radius of its curvature. The solution has no singularity when the surfaces are almost in contact (i.e. the gel layers are touching), which differs from the case of smooth surfaces. If the thickness of the porous layer tends to zero, the results reduce to those given by Cox (1974).

The hydrodynamic interaction of two particles covered with a gel layer compares with the interaction of smooth particles. The force acting on the smooth particles is calculated by Cox's formula. There are two ways of comparison: case 1 (figure 2A) the smooth particle radius is equal to the solid surface radius of the particle covered with the gel layer (e.g. the gel layer is formed by the adsorption of polymer); case 2 (figure 2B) the smooth particle radius is equal to the solid surface radius plus the gel layer thickness together (e.g. the gel layer is formed by dissolution of the particle's surface). For either case the force is expressed by the coefficient F_0 units, defined by

$$F_0 = \frac{\eta\pi(U_2^* - U_1^*)L}{(\lambda_1 + \lambda_2)\sqrt{\lambda_1\lambda_2}}.$$

Some calculated results are shown in figures 3–6 for various values of the porosity and thickness of the gel layer (curves 1 and 3 show the force acting on the particles covered with the gel layer in cases 1 and 2, respectively; curve 2 shows the force acting on the surface of the smooth particle). It is seen that the dissolution of the particle surface always decreases the force in comparison with smooth particles; on the contrary, the adsorption of polymer on the particle surface always increases the acting force. With increasing distance between the particles, the force of interaction decreases quickly. Increasing the gel layer thickness leads to a decrease in the acting force value at the point of contact and accelerates the tendency of the force to zero with distance. The force

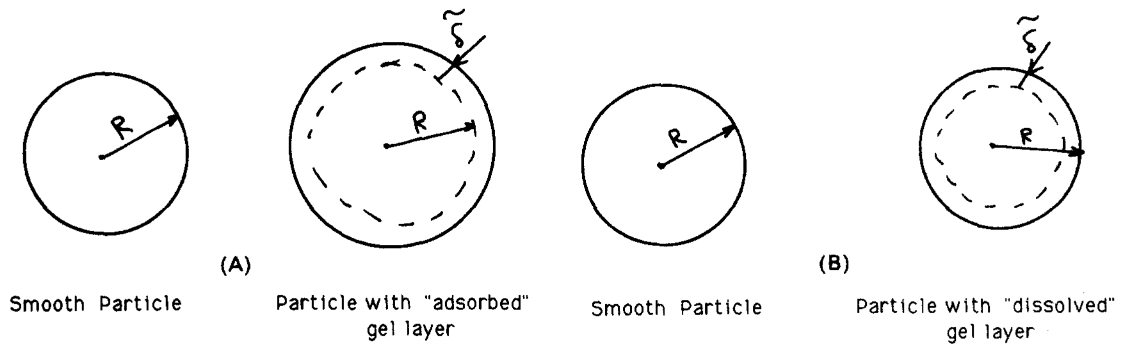

Figure 2(A). "Adsorbed" gel layers. (B) "Dissolved" gel layers.

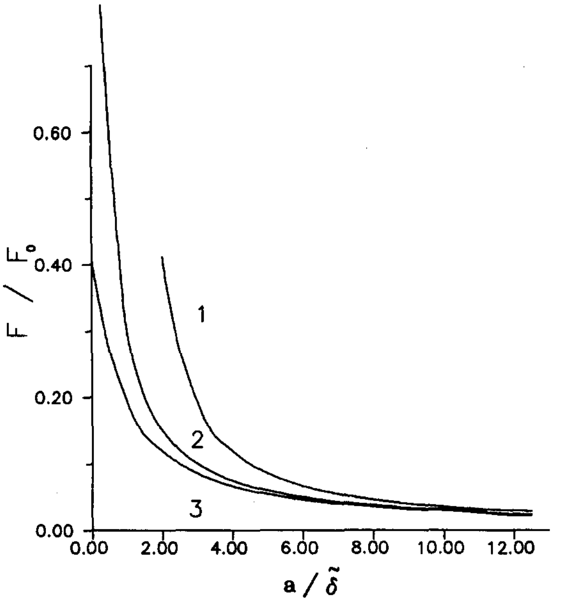

Figure 3. Dependence of relative force on the gap width. Porosity = 0.7, thickness of the gel layer = 1 nm.

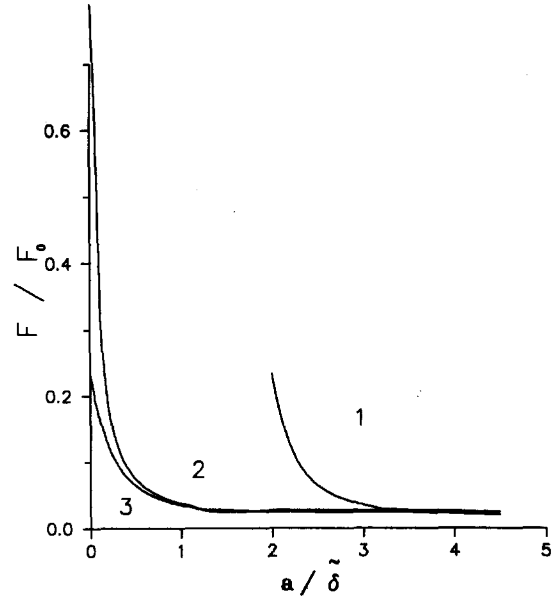

Figure 4. Dependence of relative force on the gap width. Porosity = 0.7, thickness of the gel layer = 8 nm.

of the hydrodynamic interactions approaches that of Cox for high porosities. Note that in two cases mentioned above the parameter a is the distance between the solid surfaces (case 1) and the distance between the gel layers (case 2). This fact was taken into account in calculations according to [75].

Analysis of the hydrodynamic interactions of two particles covered with a gel layer shows that utilization of Cox's formula for the interpretation of interaction force measurements and determination of the gel layer thickness in this way may lead to significant errors.

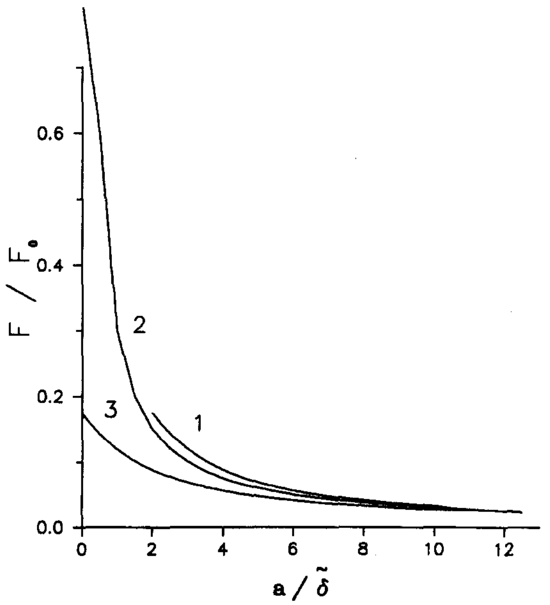

Figure 5. Dependence of relative force on the gap width. Porosity = 0.9, thickness of the gel layer = 1 nm.

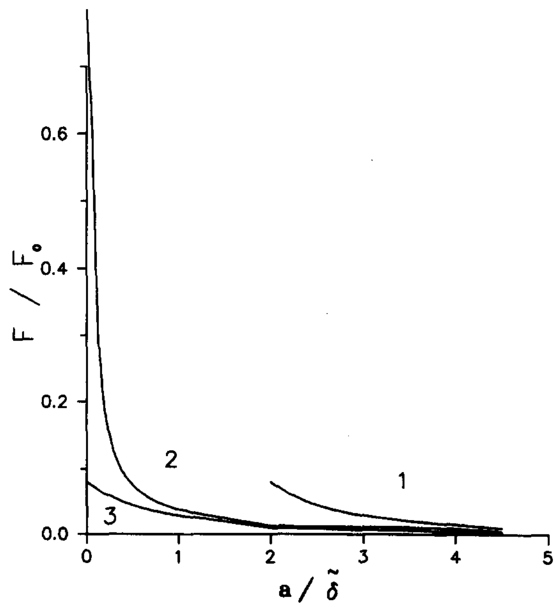

Figure 6. Dependence of relative force on the gap width. Porosity = 0.9, thickness of the gel layer = 8 nm.

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